Hopfield Neural Networks

and

Their Applications

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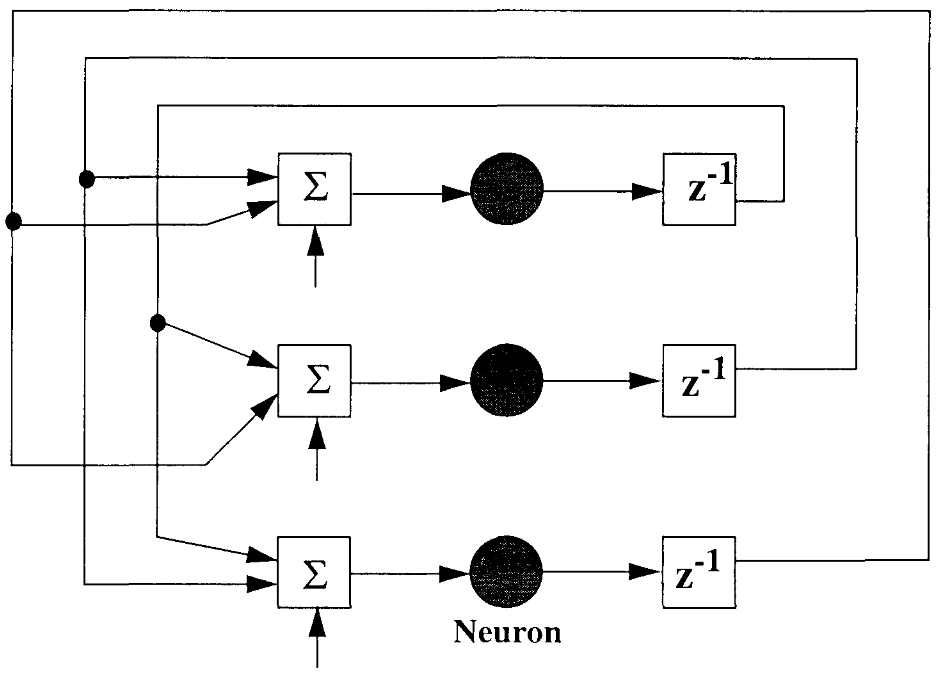
CONTENTS

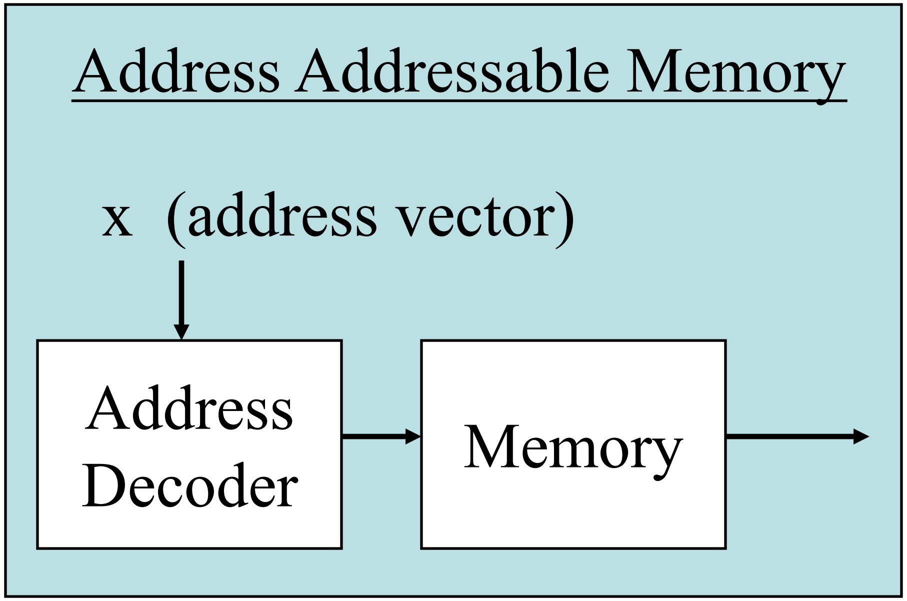
The Hopfield Neural Network (HNN)

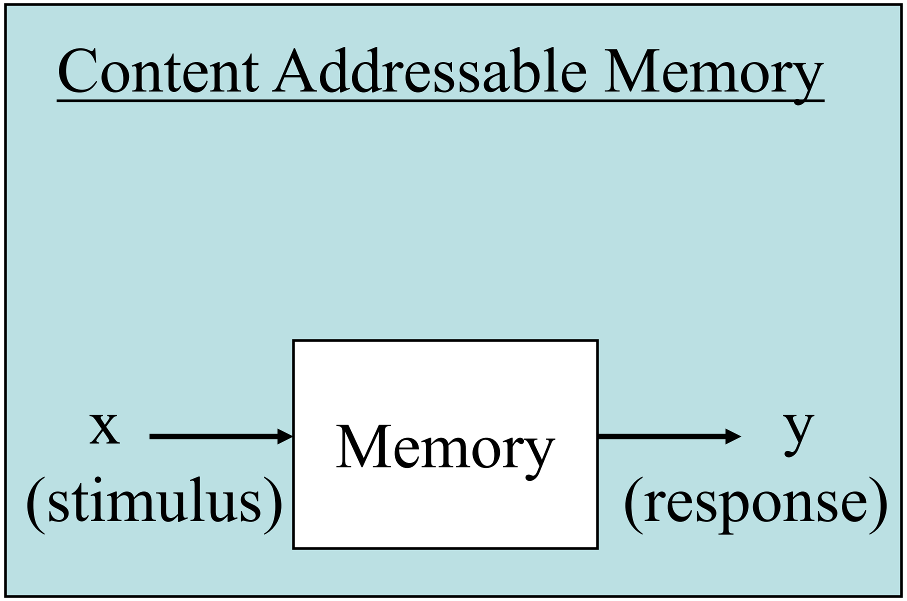
- Recurrent Neural Network
- One layer neural network with full connection



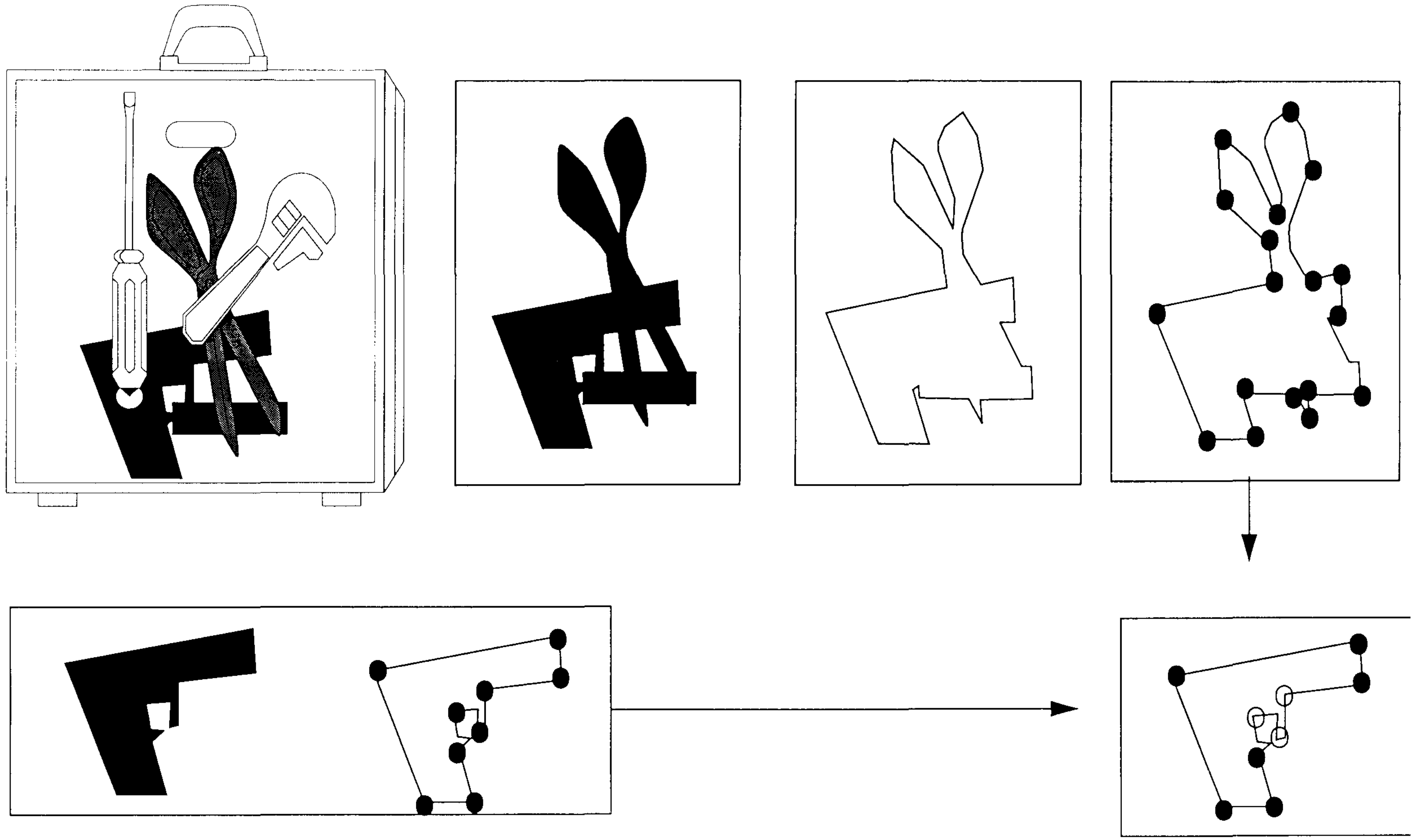
The Hopfield Neural Network

- Two aspects of Hopfield neural networks (HNN)
  - Associative memory (content addressable memory)
  - Optimization of energy function with quadratic form
- Associative memory ?



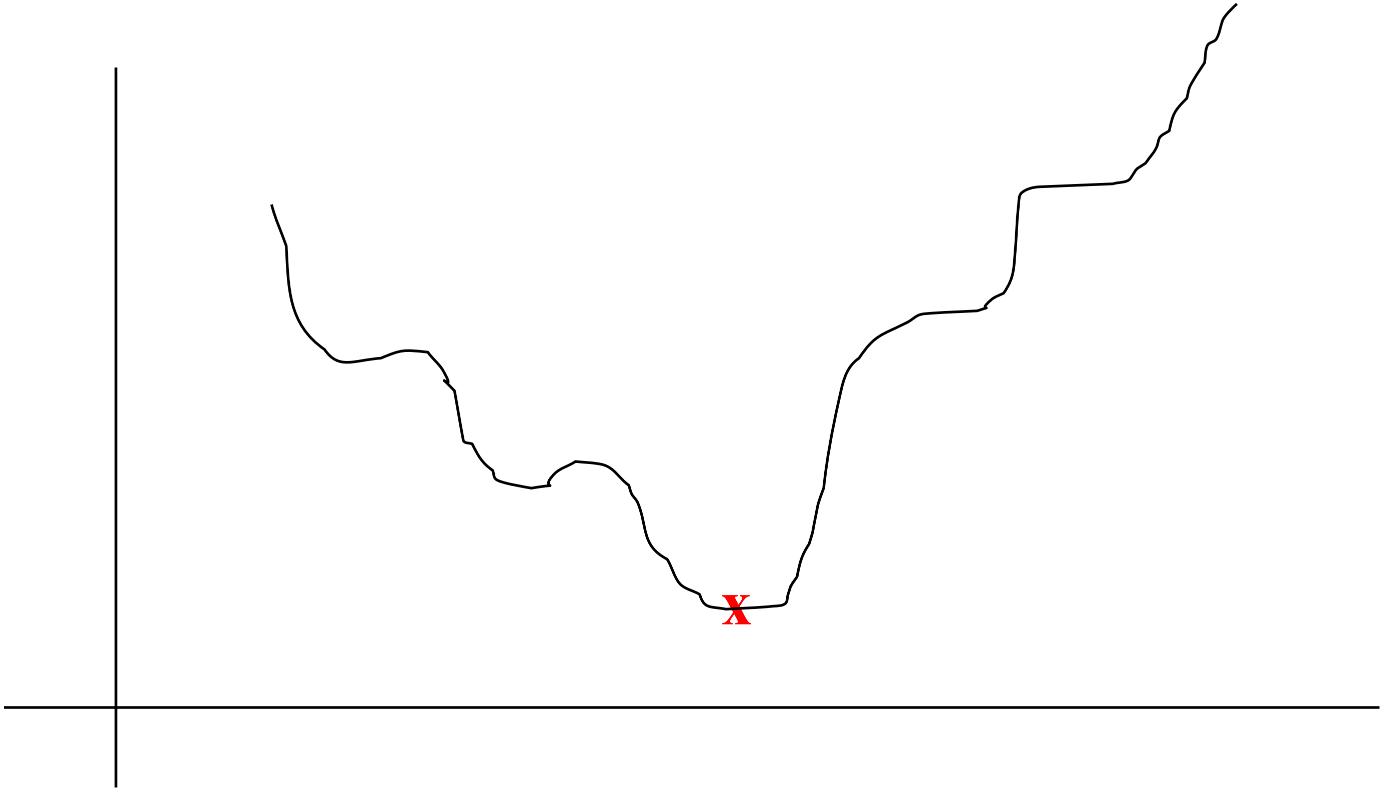


Human perception with associate memory



Optimization

Find a minimum energy function



Applications

- DSP
- Communications
- Combinatorial optimization etc.,

Associative Memory (CAM)

- Types of CAM

  - Matrix associative memory
  - Walsh associative memory
  - Network associative memory (Hopfield Neural Network)

Linear Associative Memory

- Memory Construction for associated pattern (x_k, y_k):

  $M_k = x_k * y_k^T$

- Retrival:

  $M_k * x_k = < x_k^T * x_k > * y_k$

- Overlay of patterns:

$$M = \sum_{k=1}^{P} M_k = \sum_{k=1}^{P} x_k . y_k^T$$

- Pattern Retrival:

$$M . x_i = \sum_{k=1}^{P} M_k . x_i = M_i . x_i + \sum_{k \neq i = 1}^{P} M_k . x_i$$

Walsh Associative Memory

- Walsh functions (Hadmard Matrix)

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} H_2 & H_2 \\ H_2 & \overline{H_2} \end{bmatrix}$$

- Memory construction

$$M = \mathrm{W_k} \,.\, \mathrm{x_k}^{\mathrm{T}}$$

- Retrival of Walsh Index

$$\mathrm{W_k}^{\mathrm{T}} \,.\, \mathrm{M} \,.\, \mathrm{x_k}$$

- Pattern Retrival

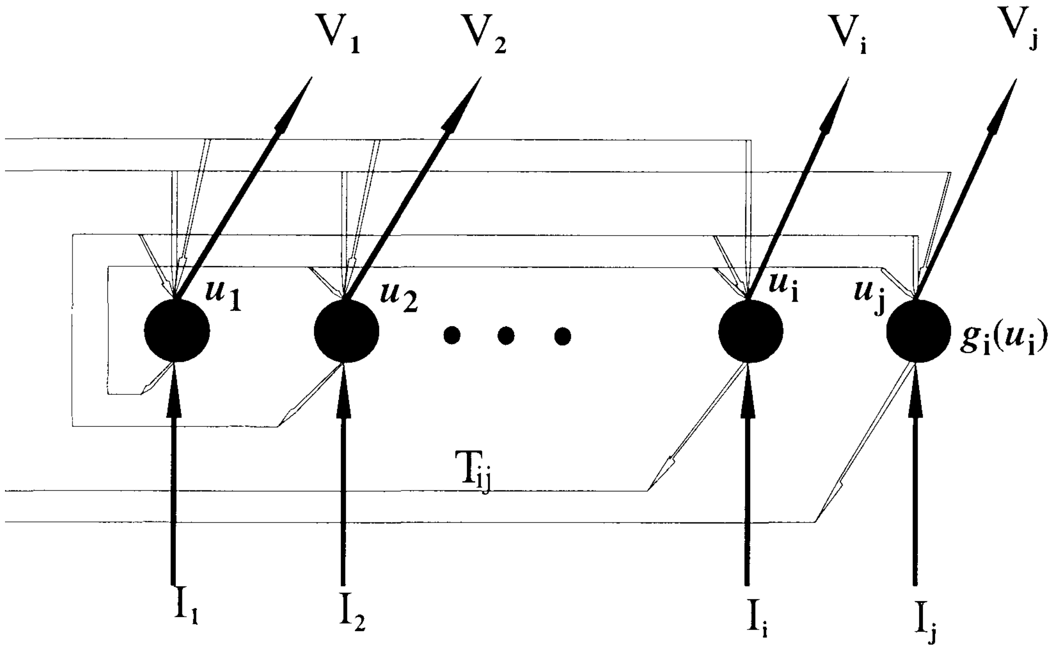
$$\mathrm{W_k}^{\mathrm{T}} \,.\, \mathrm{M}$$

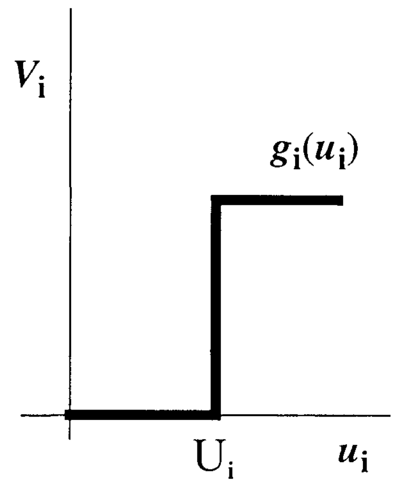
Network Associative Memory

- Features of HNN:
  - Similar to Linear Associative Memory
  - Network Modeling
  - Asynchronous processing
  - Massive distributed and parallel processing
  - Simple architecture and analog VLSI implementation

The Hopfield Neural Network

- The Architecture of the Hopfield Neural Network





The Hopfield Neural Network

- Constructing T (Information Storage):

$$T = \sum_{k=1}^{P} V^{(k)} . V^{(k)T} - pI$$

- Retrival:
  - The total input to the ith neuron:

$$u_i = \sum_{i \neq j} \sum T_{ij} V_j + I_i$$

  - Updating

$$V_i \rightarrow V_0 \text{ if } \sum_{j \neq i} T_{ij} V_j + I_i < U_i$$

$$V_i \rightarrow V_1 \text{ if } \sum_{j \neq i} T_{ij} V_j + I_i > U_i$$

- Energy function of HNN:

$$E = -\frac{1}{2} \sum_{i \neq j} \sum T_{ij} V_i V_j + \sum_i I_i V_i + \sum_i U_i V_i$$

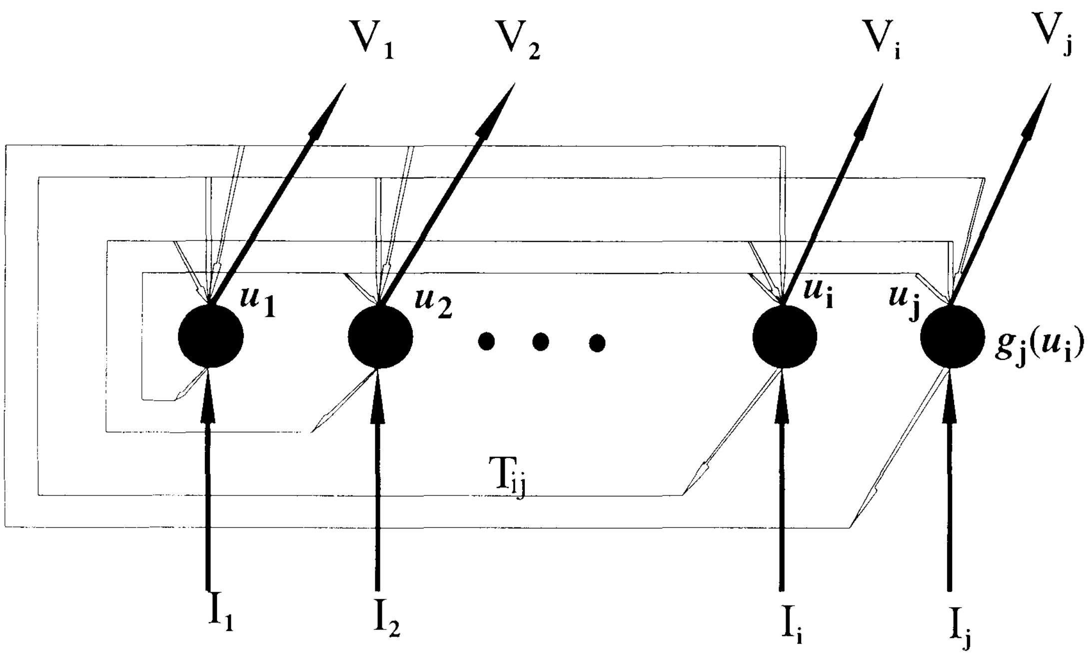
- The change ΔE in E due to changing the state of ith neuron by V_i is

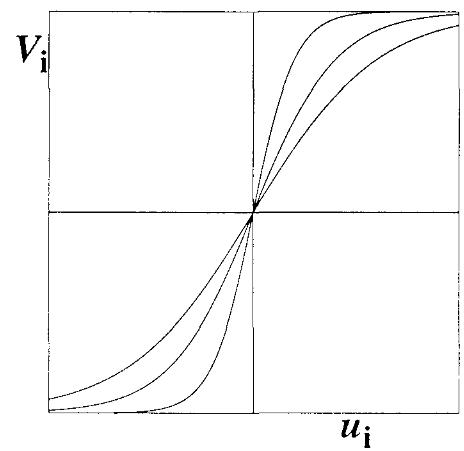
$$\Delta E = -1.\left[\sum_{k=1}^{P} T_{ij} V_j + I_i - U_i \right] \Delta V_i$$

The Continuous Hopfield Neural Network

- Energy function

$$E = -\frac{1}{2}\sum_{i\neq j}\sum T_{ij}V_iV_j - \sum_i I_iV_i + \sum_i\left(\frac{1}{R_i}\right)\int_0^{V_i} g_i^{-1}(v)\,dV$$





The Continuous Type Network

- Energy function of the Hopfield Network

$$E = -\frac{1}{2}\sum_{i \neq j}\sum T_{ij}V_iV_j - \sum_i I_iV_i + \sum_i \left(\frac{1}{R_i}\right)\int_0^{V_i} g_i^{-1}(v)dV$$

- Equation of the Motion of the Network

$$\frac{du_i}{dt} = -\frac{u_i}{R} + \sum_j T_{ij}V_j + I_i$$

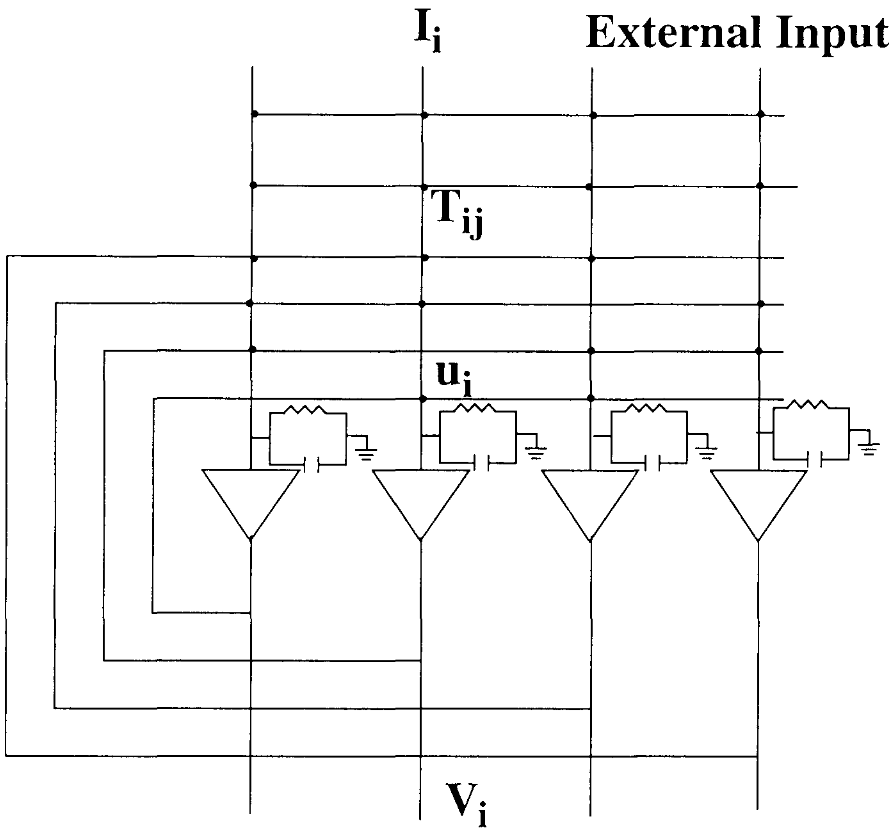
- Input-output relationship

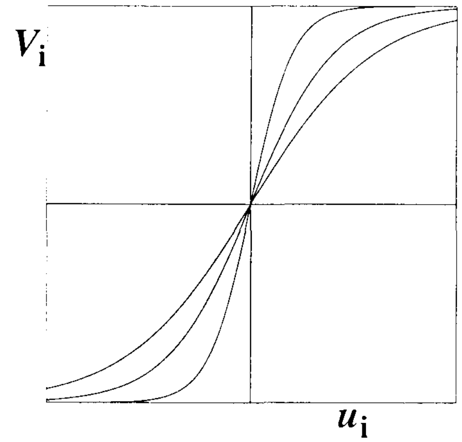
$$g(u_i) = -\frac{1}{1 + \exp(-u_i / R)}$$

- Stability of the system

$$\frac{\partial E}{\partial t} = -C_i . g_i^{-1}{}'(V_i).\left(\frac{\partial V_i}{\partial t}\right)^2$$

The Continuous Type Network





Application 1 (A/D Converter)

- 4 bit A/D Converter:

$$\sum_{i=0}^{3} V_i.2^i \approx x$$

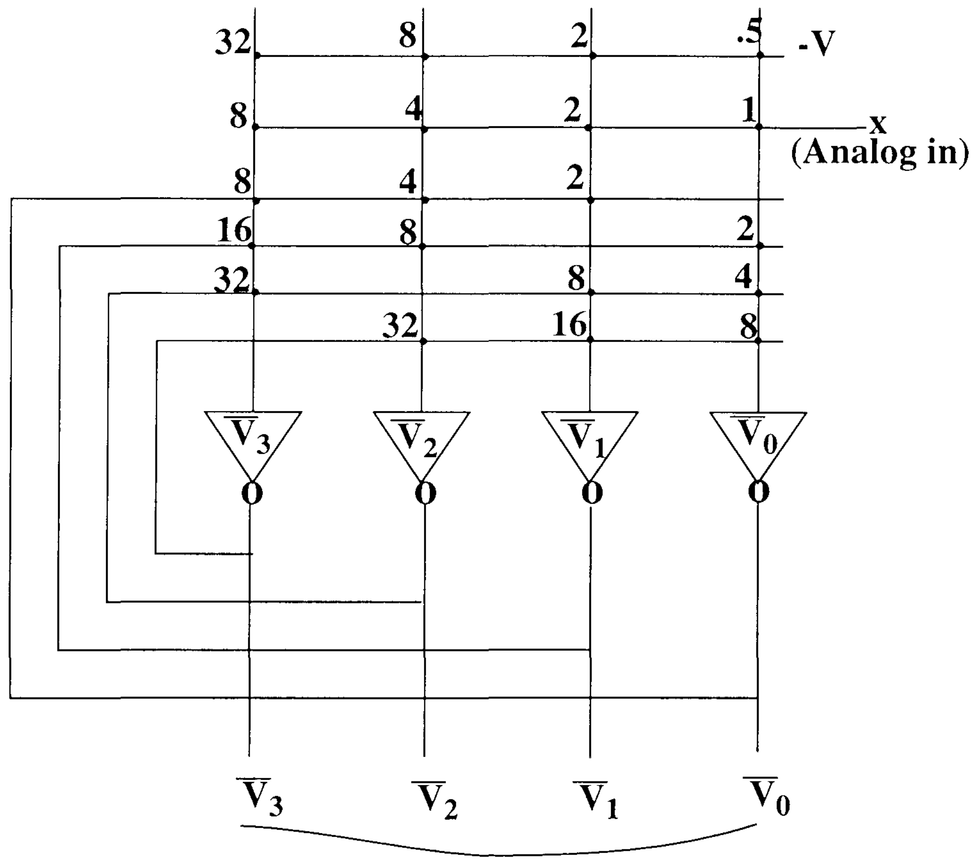
- Energy function for the A/D converter:

$$E = \left(x - \sum_{i=0}^{3} V_i.2^i \right)^2 - \sum_{i=0}^{3} (2^i)^2.[V_i.(V_i - 1)]$$

- Energy function for Hopfield network:

$$E = -\frac{1}{2} \sum_{j=0}^{3} \sum_{i \neq j=0}^{3} (-2^{i+j}) V_i V_j - \sum_{i=0}^{3} (-2^{(2i-1)} + 2^i x).V_i$$

Application 1 (A/D Converter)



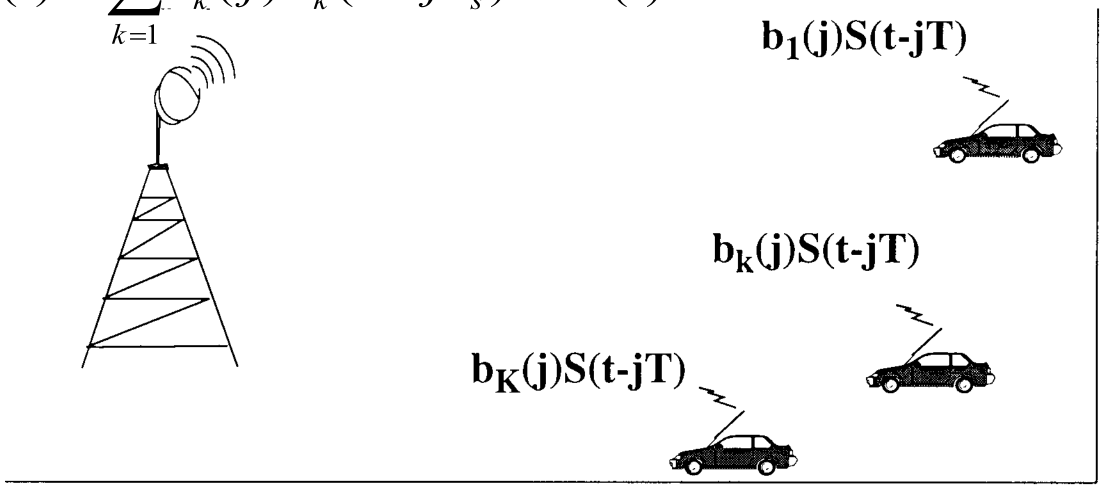
$$E = -\frac{1}{2}\sum_{j=0}^{3}\sum_{i\neq j=0}^{3}(-2^{i+j})V_iV_j - \sum_{i=0}^{3}(-2^{(2i-1)} + 2^i x).V_i$$

Application 2 (Multiuser Detection)

- Problem definition:

  Recover binary information sent from multiple transmitters

$$r(t) = \sum_{k=1}^{K} b_k(j).s_k(t - jT_s) + \sigma n(t)$$



- The receiver observes

$$r(t) = \sum_{k=1}^{K} b_k(j).s_k(t - jT_s) + \sigma n(t), \qquad t \in [jT_s, jT_s + T_s]$$

Application 2 (Multiuser Detection)

- Maximum likelihood detection:
  - Maximize likelihood probability:

$$P[r(t)|_b] = e^{\Omega(b)}$$

$$where \quad \Omega(b) = \int_0^T (r(t) - \sum_i b_i . s(t))^2 \, dt$$

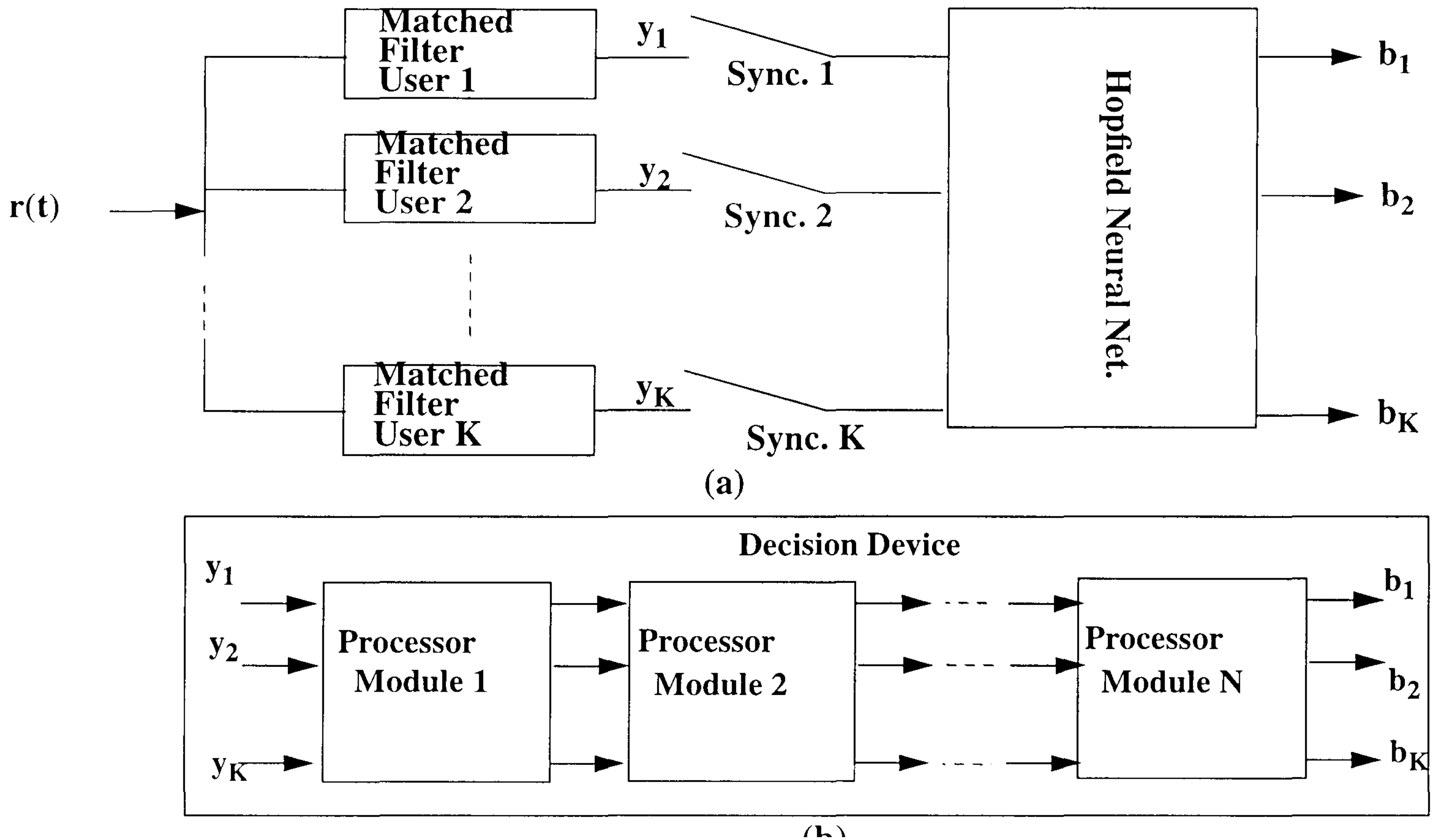
$$\hat{\mathrm{b}} \in \arg\max_{\mathrm{b} \in \{-1,1\}^K} \int_0^T \left[r(t) - \sum_{k=1}^{K} b_k s_k(t) \right]^2 dt$$

  - Further developed as follows:

$$\hat{\mathrm{b}} \in \arg\max_{\mathrm{b} \in \{-1,1\}^K} \quad 2y^T b - b^T H b$$

$$where \quad y_k = \int_0^{T_s} r(t).s_k(t)dt \quad \text{and } H_{ij} = \int_0^{T_s} s_i(t).s_j(t)dt$$

$$\hat{b} \in \arg\max_{b \in \{-1, 1\}^K} 2y^T b - b^T H b$$



Application 3 (Object Recognition)

- Problem definition:

  Find out corresponding points between those in two objects ….

- An objective function for pattern recognition:

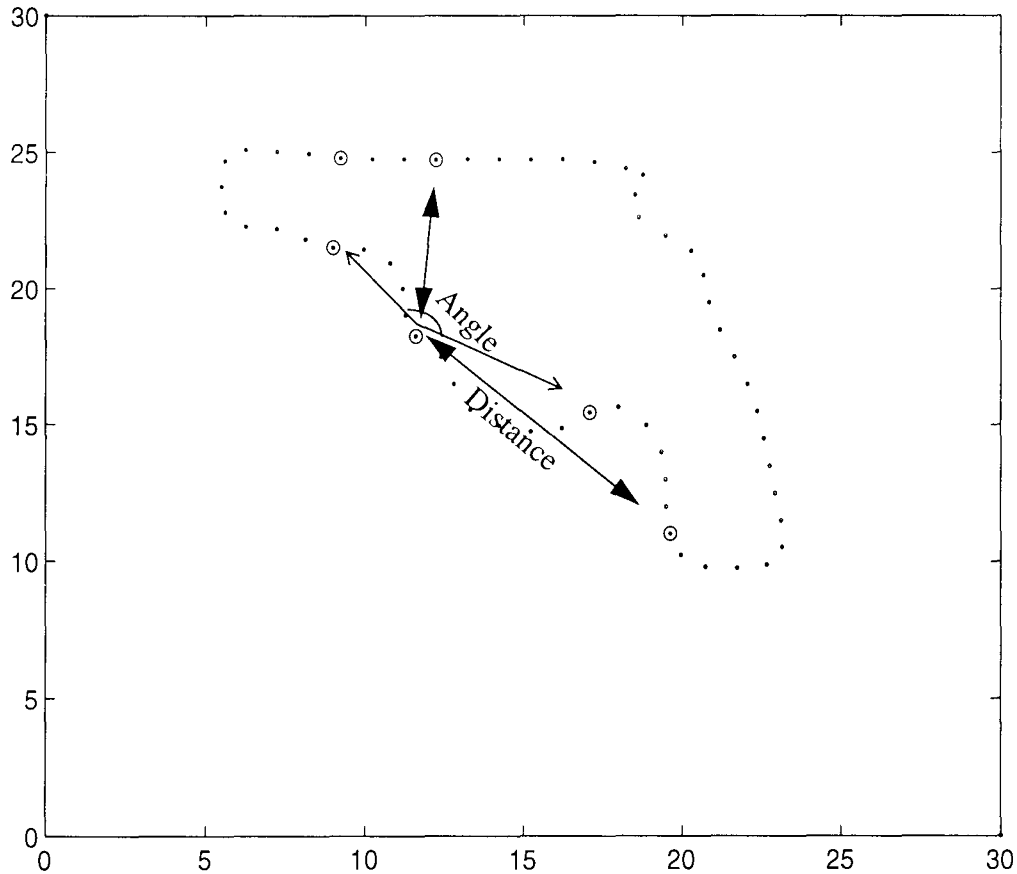
$$E = -\frac{A}{2}\sum_i \sum_j \sum_k \sum_l C_{ijkl} V_{ik} V_{jl} + \frac{q}{2}\left(\sum_i \sum_k \sum_{l \neq k} V_{ik} V_{il} + \sum_k \sum_i \sum_{j \neq i} V_{ik} V_{jk}\right)$$

- Energy function for Hopfield network:

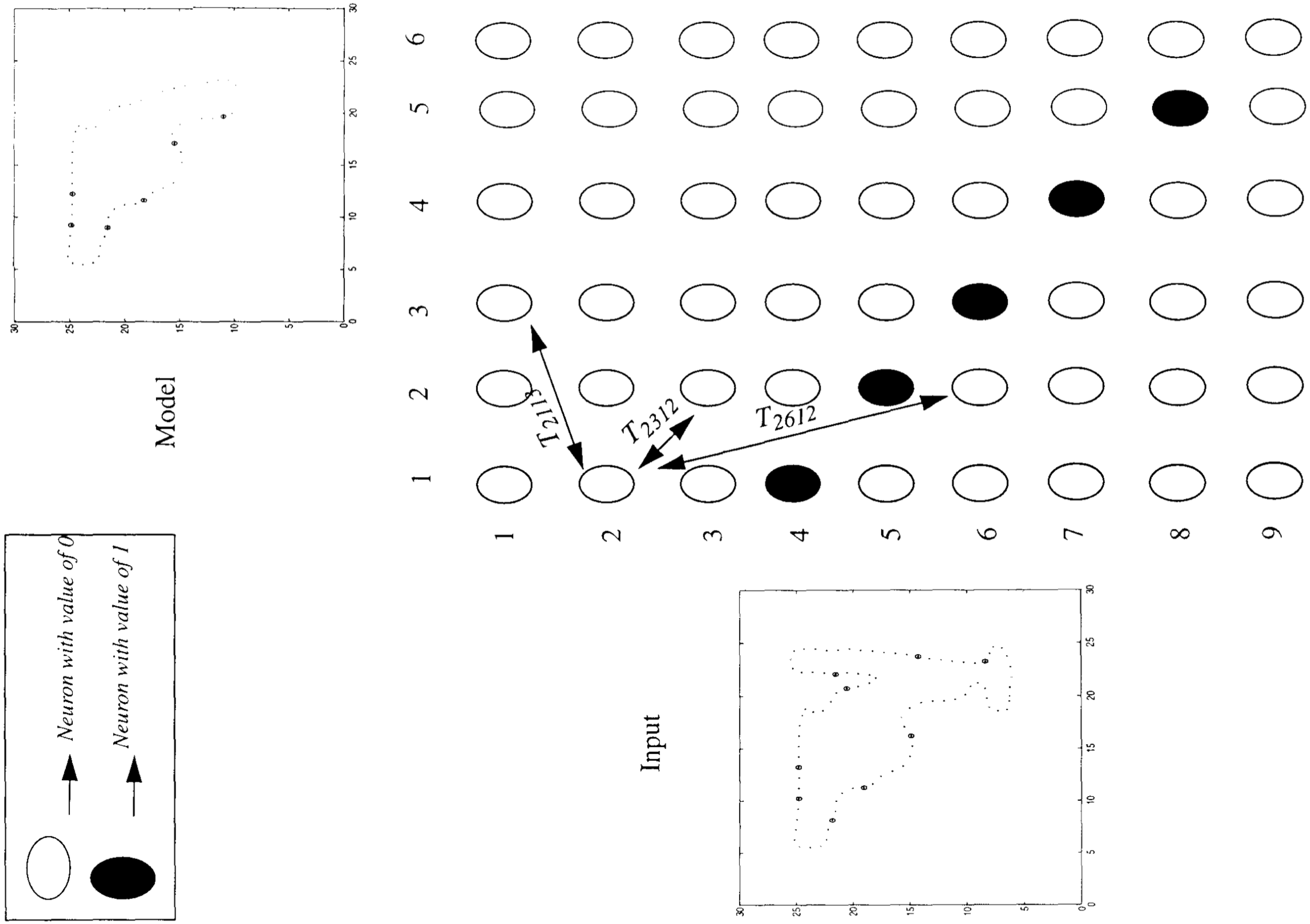
$$E = -\frac{1}{2}\sum_i \sum_j \sum_k \sum_l T_{ijkl} V_{ik} V_{jl} + \sum_i \sum_k I_{ik} V_{ik}$$

$$where \quad T_{ijkl} = AC_{ijkl} - q(\delta_{ij} + \delta_{kl}) + 2q.\delta_{ij}\delta_{kl}$$

Application 3 (Object Recognition)

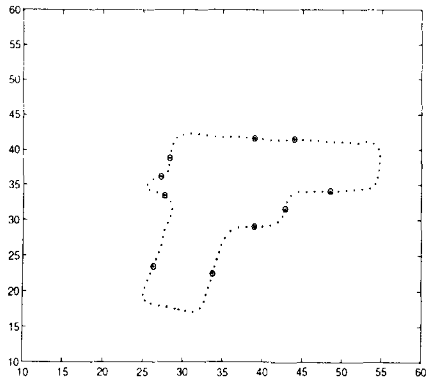


Application 3 (Object Recognition)

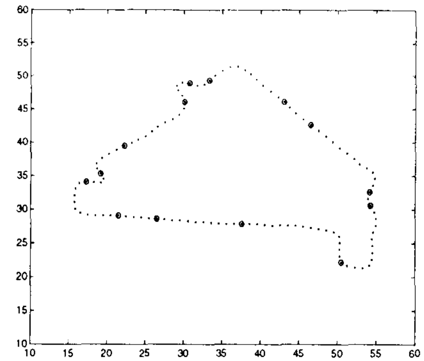


Application 3 (Object Recognition)

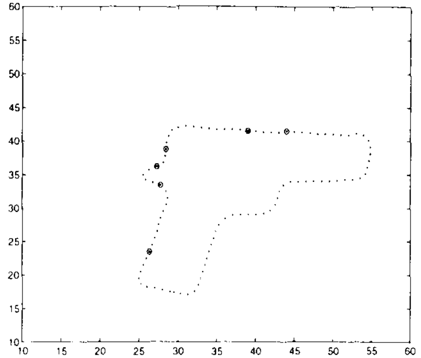
Model



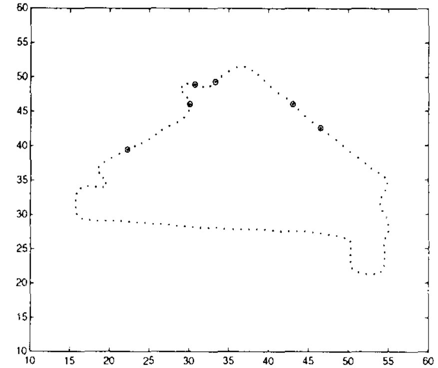
Input



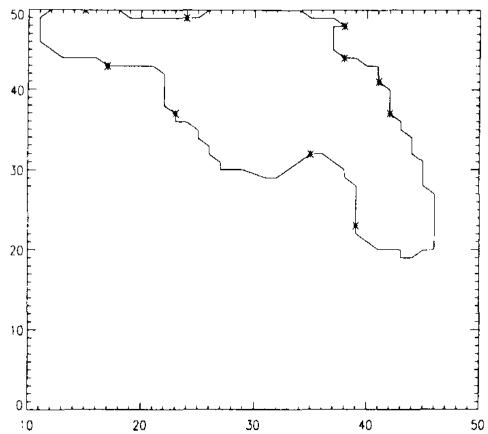
Matched Output



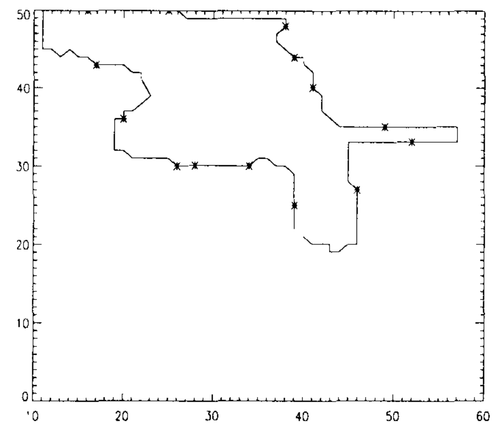
Matched Output



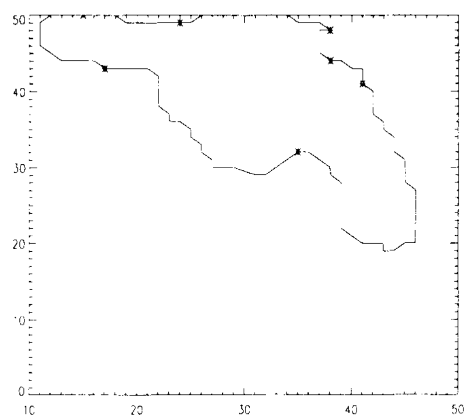
Application 3 (Object Recognition)



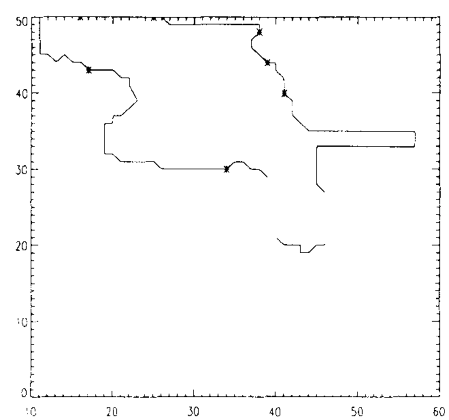
(a)



(b)



(c)



(d)